Journal of Computational and Applied Mathematics 233 (2009) 293–307

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On solution uniqueness of elliptic boundary value problems

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ARTICLE INFO

Article history:
Received 24 July 2008
Received in revised form 25 May 2009

MSC:
65N10
65N30

Keywords:
Uniqueness solution
Elliptic boundary equation
Eigenvalue problems
Finite element method
Finite difference method

ABSTRACT

In this paper, we consider the problem of solution uniqueness for the second order elliptic boundary value problem, by looking at its finite element or finite difference approximations. We derive several equivalent conditions, which are simpler and easier than the boundedness of the entries of the inverse matrix given in Yamamoto et al., [T. Yamamoto, S. Oishi, Q. Fang, Discretization principles for linear two-point boundary value problems, II, Numer. Funct. Anal. Optim. 29 (2008) 213–224]. The numerical experiments are provided to support the analysis made. Strictly speaking, the uniqueness of solution is equivalent to the existence of nonzero eigenvalues in the corresponding eigenvalue problem, and this condition should be checked by solving the corresponding eigenvalue problems. An application of the equivalent conditions is that we may discover the uniqueness simultaneously, while seeking the approximate solutions of elliptic boundary equations.

1. Main results

We first consider the second order self-adjoint elliptic boundary value problem

$$\mathcal{L}u = -\left\{\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial u}{\partial y}\right)\right\} + ru = f \quad \text{in } S, \qquad u = 0 \quad \text{on } \Gamma, \tag{1.1}$$

where S is a polygon, $\Gamma = \partial S$ is its boundary, $p = p(x, y) \geq p_0 > 0$, the functions p, r and f are smooth enough, and the sign of the function $r = r(x, y)$ is indefinite. When $r \geq 0$, the solution of (1.1) exists uniquely. Otherwise, there arises a problem of existence and uniqueness of solutions. The solution is still existent and unique if $r(<0)$ satisfies

$$|r(x, y)| < \mu_{\min}, \tag{1.2}$$

where $\mu_{\min}$ is the minimal eigenvalue of the following eigenvalue problems,

$$-\left\{\frac{\partial}{\partial x}\left(p\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial w}{\partial y}\right)\right\} = \mu w \quad \text{in } S, \quad w = 0 \quad \text{on } \Gamma. \tag{1.3}$$

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doi:10.1016/j.cam.2009.07.040

Suppose that $r(<0)$ is constant, let $r=-k^2$. If k^2 is not an eigenvalue μ_l of (1.3), Eq. (1.1) is the Helmholtz equation, and the solution of (1.1) is existent and unique. However, when $k^2=\mu_l$, the solutions may not exist, and are not unique if existing under some conditions; see Remark 2.1.

In Yamamoto [1], the error bounds are derived for two-point boundary value problems by the finite element method (FEM) and the finite difference method (FDM), under the assumption of the uniqueness solution. In the recent paper [2,3], for further application, the equivalence between the solution uniqueness and the following condition is explored,

$$|g_{ij}| \le C, \quad \text{as } h \to 0, \tag{1.4}$$

where h is the maximal diameter of quasi-uniform finite elements and difference grids, C is a constant independent of h, $\boldsymbol{A}^{-1}=(g_{i,j})$, and $\boldsymbol{A}$ is the discrete matrix resulting from the FEM and FDM. For 1D problem, since $\boldsymbol{A}$ is the tri-diagonal matrix, the explicit entries $g_{i,j}$ can be obtained in [1–3], and the condition (1.4) is easy to be examined. However, for 2D problems of elliptic equations, the condition (1.4) is complicated for real application. In this paper, we will propose new conditions of uniqueness solution[1], which are simple and easier in computation.

Consider the corresponding eigenvalue problem of (1.1),

$$\mathcal{L}w = -\left\{\frac{\partial}{\partial x}\left(p\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(p\frac{\partial w}{\partial y}\right)\right\}+rw=\lambda w \quad \text{in } S, \quad w=0 \quad \text{on } \Gamma. \tag{1.5}$$

Denote the minimal eigenvalue in magnitude

$$|\lambda_{\min}| = \min_l |\lambda_l|. \tag{1.6}$$

Then the uniqueness of solutions for (1.3) is equivalent to (see [1–3])

$$\lambda_{\min} \neq 0. \tag{1.7}$$

We will use the FEM and the FDM to approximate (1.1), and discover whether or not their solutions are unique. For simplicity, we only describe the linear and bilinear FEM, and derive the uniqueness conditions. For the FDM, the similar uniqueness conditions can be obtained easily. In Section 4, numerical experiments are carried out by both FEM and FDM.

Let S be divided into quasi-uniform triangles $\triangle_i$ and quasi-uniform rectangles $\square_j$. Then $S=S_h=(\cup_i \triangle_i)\cup\left(\cup_j \square_j\right)$. Let $\{\mathrm{P}_i\}$ be the set of grid points of elements. We denote h_i the maximal diameters of elements respectively, and h the maximal diameter of all $\triangle_i$ and $\square_j$. The elements are said to be quasi-uniform if they are regular and if $\frac{h}{\min_i h_i} \le C$, where C is a constant independent of h. Also denote by V_h^0 the set of piecewise linear and bilinear functions satisfying $v|_\Gamma=0$. The linear and bilinear FEM reads: Find $u_h \in V_h^0$ such that

$$a(u_h,v)=f(v), \quad \forall v \in V_h^0, \tag{1.8}$$

where

$$a(u,v)=\iint_S \left\{p\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)+ruv\right\}, \tag{1.9}$$

$$f(v)=\iint_S fv. \tag{1.10}$$

The equation (1.8) may be written as the system of linear algebraic equations,

$$\boldsymbol{A_h x_h}=\boldsymbol{b_h}, \tag{1.11}$$

where the matrix $\boldsymbol{A_h}\in R^{n\times n}$, $\boldsymbol{b_h}\in R^n$ is the known vector, and $\boldsymbol{x_h}\in R^n$ is the unknown vector consisting of $(u_h)_i=u_h(\mathrm{P}_i)$, where $\mathrm{P}_i\in S$. First, the error bounds of u_h can be derived if the solution of (1.1) exists uniquely. Such a uniqueness problem is equivalent to the following condition,

$$\|u_h\|_{0,S}=\sqrt{\iint_S u_h^2}\le C, \tag{1.12}$$

where C is a constant independent of h. In this paper, C is a constant independent of h but dependent on $\frac{1}{\lambda_{\min}}$ (see [4]), whose values might be different at different occurrences. Since there exists the equivalence (see Lemma 2.3),[2]

$$\|u_h\|_{0,S}\asymp O\left(\frac{1}{\sqrt{n}}\|\boldsymbol{x_h}\|_2\right), \tag{1.13}$$

[1] For simplicity, we only use the phrase uniqueness solution (or uniqueness of solutions) even if the solutions may not be unique or may not exist.

[2] The equivalence symbol, $a\asymp O(b)$, or $a\asymp b$ means that there exist two constants C_1 and C_2, which are independent of h, such that $C_1 b\le a\le C_2 b$, $b>0$.

where n is the number of the interior nodes, and $\|\boldsymbol{x_h}\|_2$ is the 2-norm defined by

$$\|\boldsymbol{x_h}\|_2 = \left\{\sum_i (u_i^h)^2\right\}^{1/2}. \tag{1.14}$$

Denote $Av2 = \frac{1}{\sqrt{n}}\|\boldsymbol{x_h}\|_2$. Hence, we may also use the other uniqueness condition

$$Av2 \le C. \tag{1.15}$$

Define the average of nodal solutions

$$Av = \frac{1}{n}\sum_i |u_i^h|. \tag{1.16}$$

We can also prove

$$Av \asymp O\left(\frac{1}{\sqrt{n}}\|\boldsymbol{x_h}\|_2\right), \tag{1.17}$$

to give the other uniqueness condition

$$Av \le C. \tag{1.18}$$

Moreover, we will also show the simplest uniqueness conditions

$$\max_i |u_i^h| \le C, \tag{1.19}$$

which was used in [2] for computation without proof. Evidently, the uniqueness conditions (1.15), (1.18) and (1.19) are much simpler than (1.4), and easier in computation.

However, when $\lambda_{\min} = 0$, $\lambda_{\min}^h \neq 0$ but $\lambda_{\min}^h \approx 0$, the above uniqueness conditions may be observed numerically only at very small h. In particular, when the function $f(x, y)$ is just or nearly orthogonal to the eigenfunction of the zero eigenvalue for the differential operator $\mathcal{L}$, the boundedness of the uniqueness conditions can be found only at the infinitesimal h. Hence, in these cases, for both theory and computation, it is suggested that the uniqueness solution should be determined by checking $\lambda_{\min} \neq 0$ of the differential operator $\mathcal{L}$, which are solicited to investigate by other highly accurate numerical algorithms.

This paper is organized as follows. In Section 2, the uniqueness conditions of the solutions are derived for self-adjoint elliptic problems by the FEM, and in Section 3, their extensions are discussed for non-self-adjoint elliptic problems. In the last section, numerical solutions are reported.

2. The finite element method

The FEM for the eigenvalue problem (1.5) in a weak form reads: Find $(\lambda_h, w_h) \in R \times V_h^0$ such that

$$a(w_h, v) = \lambda_h (w_h, v), \quad \forall v \in V_h^0, \tag{2.1}$$

where

$$a(w, v) = \iint_S \left\{ p\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial y}\right) + rwv \right\}, \tag{2.2}$$

$$(w, v) = \iint_S wv. \tag{2.3}$$

The Eq. (2.1) is also written as the generalized eigenvalue problem,

$$\boldsymbol{A_h y_h} = \lambda_h \boldsymbol{B_h y_h}. \tag{2.4}$$

where $\boldsymbol{A_h}$ is given in (1.11), the matrix $\boldsymbol{B_h} \in R^{n\times n}$ is symmetric and positive definite, and $(\lambda_h, w_h)(= (\lambda_h, \boldsymbol{y_h}))$ are the approximate eigenpairs of (λ, w). Denote the eigenvectors,

$$\boldsymbol{y_k^h} (= w_k^h) = \{\ldots, w_k^h(\mathrm{P}_i), \ldots\}. \tag{2.5}$$

They have the orthogonality

$$(w_k^h, w_\ell^h) = \langle \boldsymbol{B_h y_k^h}, \boldsymbol{y_\ell^h}\rangle = \begin{cases} 1, & k = \ell, \\ 0, & k \neq \ell, \end{cases} \tag{2.6}$$

where

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_i x_i y_i, \tag{2.7}$$

and $\boldsymbol{x} = \{\ldots, x_i, \ldots\}^T, \boldsymbol{y} = \{\ldots, y_i, \ldots\}^T$ are two vectors. Let the λ_i of (1.5) be arranged as

$$0 \le |\lambda_1| \le |\lambda_2| \le \cdots. \tag{2.8}$$

We have some lemmas.

Lemma 2.1. *For the leading eigenpairs (λ_k, w_k), there exist the error bounds of (λ_k^h, w_k^h) by the linear and bilinear FEM (2.1),*

$$|\lambda_k - \lambda_k^h| \le Ch^2, \tag{2.9}$$

$$\|w_k - w_k^h\|_{0,S} \le Ch^2, \tag{2.10}$$

where $\|w\|_{0,S} = \|w_h\|_{0,S} = 1$, and $\|w\|_{n,S}$ are the Sobolev norms.

Proof. Denote the constant $\hat{r}_{\max} = \max_S |r|$, and consider the auxiliary eigenvalue problem

$$-\left\{\frac{\partial}{\partial x}\left(p\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial w}{\partial y}\right)\right\} + (\hat{r}_{\max} + r)w = (\hat{r}_{\max} + \lambda)w = \hat{\lambda} w \text{ in } S, \tag{2.11}$$

$$w = 0 \text{ on } \Gamma. \tag{2.12}$$

For the eigenpair $(\hat{\lambda}_k, w_k)$ with $\hat{\lambda}_k \ge c_0 > 0$, where c_0 is a constant independent of h, we have (2.10) and

$$|\hat{\lambda}_k - \hat{\lambda}_k^h| \le Ch^2 \hat{\lambda}_k^2 \tag{2.13}$$

from [5–7]. Since $\hat{\lambda} = \hat{r}_{\max} + \lambda \ge c_0 > 0$, we have from $\lambda_k = \hat{\lambda}_k - \hat{r}_{\max}$,

$$|\lambda_k - \lambda_k^h| = |\hat{\lambda}_k - \hat{\lambda}_k^h| \le Ch^2 \hat{\lambda}_k^2 \le Ch^2 (\hat{r}_{\max} + |\lambda_k|)^2 \le C_1 h^2, \tag{2.14}$$

where C_1 is also a constant independent of h. This is the desired result (2.9). The inequality (2.10) is the standard result of linear and bilinear elements, and we complete the proof of Lemma 2.1. ■

Lemma 2.2. *Let $\lambda_{\min}^h = \min_k |\lambda_k^h|$ be obtained from (2.4). Assume $\lambda_{\min}^h \ne 0$, then the bound exists,*

$$\|u_h\|_{0,S} \le \frac{1}{\lambda_{\min}^h} \|f_h\|_{0,S}, \tag{2.15}$$

where u_h is the FEM solution from (1.8), and f_h is the piecewise linear and bilinear interpolation of f in (1.1).

Proof. Let $f_h (= \boldsymbol{z_h})$ be expanded as

$$f_h = \sum_{k=1}^n \alpha_k^h w_k^h, \qquad \boldsymbol{z_h} = \sum_{k=1}^n \alpha_k^h \boldsymbol{y_k^h}, \tag{2.16}$$

where $\boldsymbol{z_h} = (\ldots, (f_h)_i, \ldots)^T$, and the expansion coefficients are given by

$$\alpha_k^h = (f_h, w_k^h) = \langle \boldsymbol{B_h z_h}, \boldsymbol{y_k^h} \rangle. \tag{2.17}$$

Hence we have from (2.6)

$$\|f_h\|_{0,S}^2 = (f_h, f_h) = \left(\sum_{k=1}^n \alpha_k^h w_k^h, \sum_{k=1}^n \alpha_k^h w_k^h\right) = \sum_{k=1}^n (\alpha_k^h)^2. \tag{2.18}$$

Denote the vectors $\boldsymbol{x_h} = (\ldots, (u_h)_{ij}, \ldots)^T$ and $\boldsymbol{b_h} = \boldsymbol{B_h z_h}$. When $\lambda_{\min}^h \ne 0$, the inverse matrix $(\boldsymbol{A_h})^{-1}$ exists, and so does $(\boldsymbol{B_h}^{-1}\boldsymbol{A_h})^{-1}$. From (1.11) we have

$$\boldsymbol{x_h} = (\boldsymbol{A_h})^{-1}\boldsymbol{b_h} = (\boldsymbol{A_h})^{-1}\boldsymbol{B_h z_h} = (\boldsymbol{B_h}^{-1}\boldsymbol{A_h})^{-1}\boldsymbol{z_h}. \tag{2.19}$$

Substituting (2.16) into (2.19) and applying (2.4) yield

$$\boldsymbol{x_h} = \sum_{k=1}^n \alpha_k^h (\boldsymbol{B_h}^{-1}\boldsymbol{A_h})^{-1}\boldsymbol{y_k^h} = \sum_{k=1}^n \frac{\alpha_k^h}{\lambda_k^h}\boldsymbol{y_k^h}. \tag{2.20}$$

Hence, we have from the orthogonality (2.6) for $u_h (= \boldsymbol{x_h})$,

$$\|u_h\|_{0,S}^2 = (u_h, u_h) = \left(\sum_{k=1}^{n} \frac{\alpha_k^h}{\lambda_k^h} w_k^h, \sum_{k=1}^{n} \frac{\alpha_k^h}{\lambda_k^h} w_k^h \right) = \sum_{k=1}^{n} \frac{(\alpha_k^h)^2}{(\lambda_k^h)^2}$$
$$\leq \left(\frac{1}{\lambda_{\min}} \right)^2 \sum_{k=1}^{n} (\alpha_k^h)^2 = \left(\frac{1}{\lambda_{\min}} \right)^2 \|f_h\|_{0,S}^2, \tag{2.21}$$

where we have used (2.18). This is the desired result (2.15) and completes the proof of Lemma 2.2. ■

Denote $|\lambda_1| = |\lambda_{\min}|$ and $|\lambda_1^h| = |\lambda_{\min}^h|$. We have the following theorem.

Theorem 2.1. *When $\lambda_{\min} \neq 0$, suppose that $f \in H^2(S)$ and*

$$h \ll \sqrt{|\lambda_{\min}|}, \tag{2.22}$$

then the bound exists,

$$\|u_h\|_{0,S} \leq \frac{C}{|\lambda_{\min}|}(1 + O(h^2)). \tag{2.23}$$

When $\lambda_{\min} = 0$ but $\lambda_{\min}^h \neq 0$ satisfies

$$\left| \frac{\alpha_1^h}{\lambda_{\min}^h} \right| \gg \max_{k \geq 2} \left| \frac{\alpha_k^h}{\lambda_k^h} \right|, \tag{2.24}$$

then the asymptotics exists,

$$\|u_h\|_{0,S} \asymp O\left(\frac{1}{h^2} \right). \tag{2.25}$$

Proof. From Lemma 2.2, we have

$$\|u_h\|_{0,S} \leq \frac{1}{\lambda_{\min}^h} \|f_h\|_{0,S}, \tag{2.26}$$

where $\lambda_{\min}^h = \min_k |\lambda_k^h|$. Since there is the bound in [5,8]

$$\|f - f_h\|_{0,S} \leq Ch^2 |f|_{2,S},$$

we have

$$\|f_h\|_{0,S} \leq \|f\|_{0,S} + \|f - f_h\|_{0,S} \leq \|f\|_{0,S} + Ch^2 |f|_{2,S}. \tag{2.27}$$

Since $|\lambda_{\min}| = \min_k |\lambda_k|$, we have from (2.9)

$$\lambda_{\min}^h \geq |\lambda_{\min}| - \big|\, |\lambda_{\min}| - \lambda_{\min}^h \big| \geq |\lambda_{\min}| - Ch^2 = |\lambda_{\min}| + O(h^2). \tag{2.28}$$

Hence from (2.27) and (2.28), Eq. (2.26) leads to

$$\|u_h\|_{0,S} \leq \frac{1}{|\lambda_{\min}| + O(h^2)} (\|f\|_{0,S} + Ch^2 |f|_{2,S}). \tag{2.29}$$

When $\lambda_{\min} \neq 0$ under (2.22), then there exists the equation,

$$|\lambda_{\min}| + O(h^2) = |\lambda_{\min}| \left\{ 1 + O\left(\frac{h^2}{|\lambda_{\min}|} \right) \right\}. \tag{2.30}$$

Combining (2.29) and (2.30) yields

$$\|u_h\|_{0,S} \leq \frac{C}{|\lambda_{\min}| + O(h^2)} \leq \frac{C}{|\lambda_{\min}|} \left\{ 1 + O\left(\frac{h^2}{|\lambda_{\min}|} \right) \right\}$$
$$= \frac{C}{|\lambda_{\min}|}(1 + O(h^2)), \tag{2.31}$$

where we have used (2.22) again. This is the first result (2.23).

Next, for $\lambda_{\min} = 0$, in general $\lambda^h_{\min} \neq 0$ due to the discrete and rounding errors. We note that even if the exact value of $\lambda^h_{\min}$ for the discrete system is zero, the computed value of $\lambda^h_{\min}$ is not zero due to rounding errors. So, without loss of generality, we can deal with $\lambda^h_{\min} \neq 0$. From Lemma 2.1

$$|\lambda^h_{\min}| \asymp O(h^2). \tag{2.32}$$

Also, from (2.21), (2.24) and (2.32), we have

$$\|u_h\|^2_{0,S} = \sum_{k=1}^{n} \frac{(\alpha^h_k)^2}{(\lambda^h_k)^2} \approx \left(\frac{\alpha^h_1}{\lambda^h_1}\right)^2 \asymp O\left(\frac{(\alpha^h_1)^2}{h^4}\right). \tag{2.33}$$

This is the second result (2.25), and this completes the proof Theorem 2.1. ∎

Since the uniqueness solution is equivalent to $\lambda_{\min} \neq 0$, from Theorem 2.1 we conclude the uniqueness condition (1.12). Below, we derive the other uniqueness conditions (1.15), (1.18) and (1.19).

Lemma 2.3. *There exists the bound,*

$$\|u_h\|_{0,S} \asymp O\left(\frac{1}{\sqrt{n}}\|\boldsymbol{x_h}\|_2\right), \tag{2.34}$$

where n is the number of interior nodes of $\triangle_i$ *and* $\square_j$, *and* $\boldsymbol{x_h} = (\ldots, (u_h)_i, \ldots)^{\mathrm{T}}$ *is the solution of* (1.11).

The result is standard, so we omit the proof.

Theorem 2.2. *Let* $\lambda_{\min} \neq 0$ *and* (2.22) *be satisfied, then the conditions* (1.15), (1.18) *and* (1.19) *hold. Let* $\lambda_{\min} = 0$, $\lambda^h_{\min} \neq 0$ *and* (2.24) *be satisfied, then there exist the asymptotics as* $h \to 0$,

$$Av2 \to \infty, \qquad Av \to \infty, \qquad \max_i |u_i| \to \infty. \tag{2.35}$$

Proof. Based on Theorem 2.1 and Lemma 2.3, the uniqueness condition (1.15) is proved. The uniqueness conditions (1.18) and (1.19) can be confirmed by the following arguments. First, for $\lambda_{\min} \neq 0$ we have from the Schwarz inequality and Lemma 2.3

$$Av = \frac{1}{n}\sum_i |u^h_i| \leq \frac{1}{n}\sqrt{n}\sqrt{\sum_i (u^h_i)^2} = \frac{1}{\sqrt{n}}\|\boldsymbol{x}_h\|_2 \leq C. \tag{2.36}$$

On the other hand, for $\lambda_{\min} = 0$, we have from (2.20), (2.24) and (2.32),

$$\boldsymbol{x_h} \approx \frac{\alpha_1}{\lambda^h_{\min}}\boldsymbol{y^h_{min}} \asymp O\left(\frac{\alpha_1 \boldsymbol{y^h_{min}}}{h^2}\right), \tag{2.37}$$

where $\boldsymbol{y^h_{min}}$ is the eigenvector of (2.4) corresponding to $\lambda^h_{\min}$. The Eq. (2.37) leads to

$$Av = O\left(\frac{1}{h^2}\right), \qquad Av2 = O\left(\frac{1}{h^2}\right). \tag{2.38}$$

Based on Lemma 2.3, (2.36) and (2.38), the uniqueness condition (1.18) is proved.

Finally, we show (1.19). For $\lambda_{\min} \neq 0$ and $h \to 0$, we conclude from Lemma 2.1 that $\lambda^h_{\min} \neq 0$. The solution vector $\boldsymbol{x_h}$ is bounded from (2.20), and then all u_i are bounded. On the other hand, when $\lambda_{\min} = 0$ and $\lambda^h_{\min} \neq 0$, Eq. (2.37) under (2.24) leads to

$$\max_i |u^h_i| \asymp O\left(\frac{1}{h^2}\right). \tag{2.39}$$

This confirms the uniqueness condition (1.19). The proof of Theorem 2.2 is completed. ∎

Remark 2.1. Let us consider the solutions of (1.1) under $\lambda_{\min} = 0$. Denote the eigenpairs (λ_k, w_k) of (1.5), and $\lambda_{\min} = \lambda_1 = 0$, $w_{\min} = w_1$. When the function f is orthogonal to $w_{\min}$,

$$(f, w_{\min}) = 0, \tag{2.40}$$

by the same arguments in Lemma 2.2, we find the many solutions of (1.1)

$$u = c\, w_{\min} + \sum_{k=2}^{\infty} \frac{\alpha_k}{\lambda_k} w_k, \tag{2.41}$$

where c is an arbitrary constant, $\lambda_k (k \geq 2) \neq 0$, and $\alpha_k = (f, w_k)$. However, the solutions of (1.1) do not exist if the orthogonality condition (2.40) is violated, or if the non-homogeneous Dirichlet boundary condition $u|_\Gamma \neq 0$ is given for (1.1).

Corollary 2.1. *For $h \to 0$, conditions (1.15), (1.18) and (1.19) are equivalent to the uniqueness solution of (1.1).*

Proof. From Theorem 2.2 and Remark 2.1, there always exists the nonzero coefficient,

$$\alpha_1^h = (f_h, w_1^h) = \langle \boldsymbol{B_h z_h}, \boldsymbol{y_1^h} \rangle \neq 0, \tag{2.42}$$

due to rounding errors. Hence for $\lambda_{\min} = 0$ and $\lambda_{\min}^h \neq 0$, Eq. (2.24) must hold. This completes the proof of Corollary 2.1. ■

Remark 2.2. It is crucial that the validity of equivalent conditions (1.15), (1.18) and (1.19) is guaranteed only when h is small enough. The assumption (2.24), in which the threshold on h is given in terms of the smallest absolute value of the eigenvalues, depends on concrete practical problems. Refer also to the final remarks in the last part of Section 4.

3. Extensions to non-self-adjoint elliptic problems

Now we consider the second order non-self-adjoint elliptic boundary value problem

$$\bar{\mathcal{L}}u = -\left\{ \frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial u}{\partial y}\right)\right\} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + ru = f \quad \text{in } S,\ u = 0 \text{ on } \Gamma, \tag{3.1}$$

where the functions a and b are smooth enough, and the sign of the function $r = r(x, y)$ is still indefinite. Hence, there also arises a problem of the uniqueness of solutions. The corresponding eigenvalue problem of (3.1) is given by

$$\bar{\mathcal{L}}w = -\left\{ \frac{\partial}{\partial x}\left(p\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial w}{\partial y}\right)\right\} + a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} + rw = \bar{\lambda} w \quad \text{in } S,\ w = 0 \text{ on } \Gamma, \tag{3.2}$$

where the eigenvalues $\bar{\lambda}$ of (3.2) are complex, in general. Then the uniqueness of solutions for (3.1) is equivalent to

$$|\bar{\lambda}_{\min}| = \min_l |\bar{\lambda}_l| \neq 0, \quad \text{i.e., } \{\mathrm{Re}(\bar{\lambda}_{\min})\}^2 + \{\mathrm{Im}(\bar{\lambda}_{\min})\}^2 > 0. \tag{3.3}$$

We only use the FEM to approximate (3.1). The linear and bilinear FEM reads: Find $u_h \in V_h^0$ such that

$$\bar{a}(u_h, v) = f(v), \quad \forall v \in V_h^0, \tag{3.4}$$

where

$$\bar{a}(u, v) = \iint_S p\left\{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right\} + \left(a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + ru\right)v, \tag{3.5}$$

$$f(v) = \iint_S fv. \tag{3.6}$$

The Eq. (3.4) may be written as the system of linear algebraic equations:

$$\boldsymbol{\bar{A}_h x_h} = \boldsymbol{B_h z_h}, \tag{3.7}$$

where $\boldsymbol{x_h}$ is given in (1.11), and $\boldsymbol{z_h} \in R^n$ is the known vector consisting of $(f_h)_i = f(\mathrm{P}_i)$. Note that the matrix $\boldsymbol{\bar{A}_h} \in R^{n\times n}$ is nonsymmetric, but the matrix $\boldsymbol{B_h} \in R^{n\times n}$ is still symmetric and positive definite.

The FEM for the eigenvalue problem (3.2) in a weak form reads: Find $(\bar{\lambda}_h, w_h) \in R \times V_h^0$ such that

$$\bar{a}(w_h, v) = \bar{\lambda}_h(w_h, v), \quad \forall v \in V_h^0, \tag{3.8}$$

where

$$\bar{a}(w, v) = \iint_S p\left\{\frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial y}\right\} + \left(a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} + rw\right)v. \tag{3.9}$$

The Eq. (3.8) is also written as the generalized eigenvalue problem,

$$\boldsymbol{\bar{A}_h y_h} = \bar{\lambda}_h \boldsymbol{B_h y_h}, \tag{3.10}$$

where $\boldsymbol{\bar{A}_h}$ and $\boldsymbol{B_h}$ are given in (3.7).

Let the eigenvalues $\bar{\lambda}_i$ of (3.2) also be arranged as

$$0 \leq |\bar{\lambda}_1| \leq |\bar{\lambda}_2| \leq \cdots. \tag{3.11}$$

There exists the bound for the leading eigenvalues,

$$|\bar{\lambda}_k - \bar{\lambda}_k^h| \le Ch^\alpha, \quad 0 < \alpha \le 2, \tag{3.12}$$

where $\bar{\lambda}_k^h$ are eigenvalues of (3.10). Eq. (3.7) can be written as

$$\boldsymbol{F_h x_h} = \boldsymbol{z_h}, \tag{3.13}$$

where

$$\boldsymbol{F_h} = (\boldsymbol{B_h})^{-1} \bar{\boldsymbol{A}}_{\boldsymbol{h}}. \tag{3.14}$$

First, we have some lemmas.

Lemma 3.1. *For (3.7) the bound exists,*

$$\|\boldsymbol{x_h}\|_2 \le \frac{1}{\sigma_{\min}^h} \|\boldsymbol{z_h}\|_2, \tag{3.15}$$

where $\sigma_{\min}^h (> 0)$ is the minimal singular value of the matrix $\boldsymbol{F_h}$.

Proof. Let matrix $\boldsymbol{F_h}$ be decomposed by the singular value decomposition

$$\boldsymbol{F_h} = \boldsymbol{U} \boldsymbol{\Sigma_h} \boldsymbol{V}^{\mathrm{T}}, \tag{3.16}$$

where matrices $\boldsymbol{U} \in R^{n\times n}$ and $\boldsymbol{V} \in R^{n\times n}$ are orthogonal, and matrix $\boldsymbol{\Sigma_h} \in R^{n\times n}$ is diagonal with the positive singular values σ_i in the order: $0 < \sigma_1 \le \sigma_2 \le \cdots \le \sigma_n$. Denote

$$\boldsymbol{U} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n), \qquad \boldsymbol{V} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n), \tag{3.17}$$

we have the expansions

$$\boldsymbol{z_h} = \sum_{i=1}^n \beta_i \boldsymbol{u}_i,$$

where the expansion coefficients are given by

$$\beta_i = \boldsymbol{u}_i^{\mathrm{T}} \boldsymbol{z_h}. \tag{3.18}$$

Hence we have

$$\|\boldsymbol{z_h}\|_2 = \sqrt{\sum_{i=1}^n \beta_i^2}. \tag{3.19}$$

Since the inverse matrix $\boldsymbol{\Sigma_h}^{-1}$ of $\boldsymbol{\Sigma_h}$ is diagonal with the entries $\frac{1}{\sigma_i}$, the inverse matrix of $\boldsymbol{F_h}$ is given by

$$\boldsymbol{F_h}^{-1} = \boldsymbol{V} \boldsymbol{\Sigma_h}^{-1} \boldsymbol{U}^{\mathrm{T}}, \tag{3.20}$$

and the solution is obtained by

$$\boldsymbol{x_h} = \boldsymbol{F_h}^{-1} \boldsymbol{b} = \boldsymbol{V} \Sigma_{\boldsymbol{h}}^{-1} \boldsymbol{U}^{\mathrm{T}} \boldsymbol{z_h} = \sum_{i=1}^n \frac{\beta_i}{\sigma_i} \boldsymbol{v}_i, \tag{3.21}$$

and since $\boldsymbol{U}$ is orthogonal, we obtain from (3.19) and (3.21)

$$\|\boldsymbol{x_h}\|_2 \le \frac{1}{\sigma_1} \sqrt{\sum_{i=1}^n \beta_i^2} = \frac{\|\boldsymbol{z_h}\|_2}{\sigma_1} = \frac{\|\boldsymbol{z_h}\|_2}{\sigma_{\min}^h}, \tag{3.22}$$

where $\sigma_1 = \sigma_{\min}^h$. This is the desired result (3.15), and completes the proof of Lemma 3.1. ■

Lemma 3.2. *Let $\sigma^h_{\min} > 0$ and $f \in C(S)$. The following bounds hold:*

$$Av2 \le \frac{C}{\sigma^h_{\min}}, \tag{3.23}$$

$$Av \le \frac{C}{\sigma^h_{\min}}, \tag{3.24}$$

$$\max_i |u_i| \le \frac{C}{\sigma^h_{\min}}. \tag{3.25}$$

Proof. Since $f \in C(S)$ we have $|f_i| \le C,\ \forall i$, which gives

$$\|\boldsymbol{z_h}\|_2 = \sqrt{\sum_i f_i^2} \le C\sqrt{n}. \tag{3.26}$$

From Lemma 3.1 and (3.26) we have

$$Av2 = \frac{1}{\sqrt{n}}\|\boldsymbol{x_h}\|_2 \le \frac{C}{\sigma^h_{\min}}. \tag{3.27}$$

This is the first result (3.23), and the proof for (3.24) is similar. From (3.21) we have

$$\boldsymbol{x_h} = \frac{\beta_1}{\sigma^h_{\min}}\boldsymbol{v}_1 + \sum_{i=2}^{n}\frac{\beta_i}{\sigma_i}\boldsymbol{v}_i \approx \frac{\beta_1}{\sigma^h_{\min}}\boldsymbol{v}_1. \tag{3.28}$$

The last result (3.25) follows, and this completes the proof of Lemma 3.2. ∎

Let $|\bar{\lambda}^h_{\min}| = \min_l |\bar{\lambda}^h_l|$ be obtained from (3.10). When condition (3.3) holds, for the small enough h, we have from (3.12)

$$|\bar{\lambda}^h_{\min}| \ne 0, \tag{3.29}$$

which implies that $\bar{\lambda}^h_{\min}$ is the nonzero leading eigenvalue.

Below, let us explore the relation between $|\bar{\lambda}^h_{\min}|$ and $\sigma^h_{\min}$ of matrix $\boldsymbol{F_h}$.

Lemma 3.3. *The two conditions:*

$$|\bar{\lambda}^h_{\min}| \ne 0 \quad \text{and} \quad \sigma^h_{\min} > 0 \tag{3.30}$$

are equivalent to each other. Also, under the assumptions of (3.3) and h small enough, the bound exists,

$$0 < \sigma^h_{\min} \le |\bar{\lambda}^h_{\min}|. \tag{3.31}$$

Proof. Eq. (3.30) follows from

$$\prod_{i=1}^{n}\sigma_i = \prod_{i=1}^{n}|\bar{\lambda}^h_i|, \tag{3.32}$$

where σ_i and $\bar{\lambda}^h_i$ are the singular values and the eigenvalues of matrix $\boldsymbol{F_h}$, respectively. The proof of (3.32) is given later.

Under the assumptions, we have $|\bar{\lambda}^h_{\min}| \ne 0$, and then

$$\sigma^h_{\min} > 0. \tag{3.33}$$

Since for a matrix $\boldsymbol{B} \in R^{n\times n}$, the spectral radius is always the lower bound of all matrix norms, we have

$$|\lambda_{\max}(\boldsymbol{B})| = \rho(\boldsymbol{B}) \le \|\boldsymbol{B}\|_2 = \sigma_{\max}(\boldsymbol{B}). \tag{3.34}$$

Letting $\boldsymbol{B} = \boldsymbol{F_h}^{-1}$, we obtain

$$\frac{1}{|\lambda_{\min}(\boldsymbol{F_h})|} = |\lambda_{\max}(\boldsymbol{F_h}^{-1})| \le \|\boldsymbol{F_h}^{-1}\|_2 = \frac{1}{\sigma_{\min}(\boldsymbol{F_h})}, \tag{3.35}$$

which yields

$$\sigma^h_{\min} \le |\bar{\lambda}^h_{\min}|. \tag{3.36}$$

Combining (3.33) and (3.36) gives the second result (3.31).

Now let us show (3.32). For matrix $\boldsymbol{F_h}$, the equalities of determinants exist:

$$\det \boldsymbol{F_h} = \det \boldsymbol{F_h}^{\mathrm{T}} = \prod_{i=1}^{n} \bar{\lambda}_i^h. \tag{3.37}$$

Since $\det \boldsymbol{F_h}$ is real, we have

$$\det(\boldsymbol{F_h}^{\mathrm{T}}\boldsymbol{F_h}) = (\det \boldsymbol{F_h}^{\mathrm{T}})^2 = \left(\prod_{i=1}^{n} |\bar{\lambda}_i^h|\right)^2. \tag{3.38}$$

On the other hand, we have from (3.16)

$$\boldsymbol{F_h}^{\mathrm{T}} = \boldsymbol{V}\Sigma_{\boldsymbol{h}}\boldsymbol{U}^{\mathrm{T}}, \tag{3.39}$$

and then

$$\boldsymbol{F_h}^{\mathrm{T}}\boldsymbol{F_h} = \boldsymbol{V}\boldsymbol{\Sigma_h}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Sigma_h}\boldsymbol{V}^{\mathrm{T}} = \boldsymbol{V}\boldsymbol{\Sigma_h^2}\boldsymbol{V}^{\mathrm{T}}. \tag{3.40}$$

The determinant is given by

$$\det(\boldsymbol{F_h}^{\mathrm{T}}\boldsymbol{F_h}) = (\det \boldsymbol{V})^2 \times \det(\boldsymbol{\Sigma_h})^2 = \prod_{i=1}^{n} \sigma_i^2, \tag{3.41}$$

where we have used $\det \boldsymbol{V} = \pm 1$ for the orthogonal matrix $\boldsymbol{V}$. Combining (3.38) and (3.41) gives the desired result (3.32), and this completes the proof of Lemma 3.3. ■

Theorem 3.1. *Let conditions (1.15), (1.18) and (1.19) be satisfied for $h \to 0$, then the uniqueness solution of (3.1) exists. Suppose that the FEM and the FDM are the convergent schemes and that the condition (2.35) holds, then the uniqueness solution of (3.1) does not exist.*

Proof. Since for $h \to 0$, the conditions (1.15), (1.18) and (1.19) hold, we conclude that $\sigma_{\min}^h \geq c_0 > 0$ from Lemma 3.3, where c_0 is a constant independent of h, and that $|\bar{\lambda}_{\min}^h| \geq c_0$ from (3.31). From (3.12) we have

$$|\bar{\lambda}_{\min} - \bar{\lambda}_{\min}^h| \leq Ch^{\alpha}, \quad 0 < \alpha \leq 2. \tag{3.42}$$

Let the small h satisfy $Ch^{\alpha} \leq \frac{c_0}{2}$, then the bound exists:

$$|\bar{\lambda}_{\min}| \geq |\bar{\lambda}_{\min}^h| - Ch^{\alpha} \geq c_0 - Ch^{\alpha} \geq \frac{c_0}{2}. \tag{3.43}$$

This indicates that the uniqueness condition (3.3) holds. Consequently, conditions (1.15), (1.18) and (1.19) are the sufficient conditions of the solution uniqueness.

Next, suppose that $Av2, Av$ and $\max_i |u_i|$ are unbounded for $h \to 0$. Then we conclude that $|\bar{\lambda}_{\min}| = 0$ by the contradiction. We assume $|\bar{\lambda}_{\min}| \neq 0$ contrarily. Under the assumption, the convergent solutions of the FEM and the FEM must be bounded, and so are $Av2, Av$ and $\max_i |u_i|$, which indicates $|\bar{\lambda}_{\min}| = 0$. Consequently, under the assumption, conditions (1.15), (1.18) and (1.19) are also the necessary conditions of the uniqueness solution. This completes the proof of Theorem 3.1. ■

From Theorem 3.1 we have the following corollary.

Corollary 3.1. *Suppose that the FEM and the FDM are the convergent schemes. For $h \to 0$, conditions (1.15), (1.18) and (1.19) are equivalent to the uniqueness of solutions of (3.1).*

Remark 3.1. Denote $\boldsymbol{F_h}^{-1} = (g_{i,j})$, where $\boldsymbol{F_h}$ is given in (3.16). In [2], the condition (1.4) is equivalent to the uniqueness solution for linear two-point boundary value. In this remark, we will extend it to the elliptic equations of 2D with a brief argument. From (3.20) we have

$$\boldsymbol{F_h}^{-1} = \sum_{k=1}^{n} \frac{1}{\sigma_k} \boldsymbol{v}_k \boldsymbol{u}_k^{\mathrm{T}}, \tag{3.44}$$

where the vectors $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are given in (3.17). The entries $g_{i,j}$ of matrix $\boldsymbol{F_h}^{-1}$ are obtained as

$$g_{i,j} = \sum_{k=1}^{n} \frac{1}{\sigma_k} v_{ki} u_{kj}, \tag{3.45}$$

Table 4.1
The numerical results by the FDM with $z_p = \pi$ and $f = 1$.

N	Av	$Av2$	$u_h(0, 0.5)$	$u_h(0.5, 0.5)$	$\lambda^h_{\min}$	$\lambda^h_{\max}$	Cond
2	20.3	17.8	25.0	17.9	0.118	3.37	28.5
4	4.51	4.46	7.48	5.54	0.121	6.69	55.2
8	3.33	3.61	6.50	4.85	0.465(−1)	7.66	210
16	2.96	3.38	6.30	4.71	0.950(−2)	7.90	831
32	2.81	3.30	6.25	4.68	0.240(−2)	7.94	0.331(4)

where $\boldsymbol{u}_k = (u_{k1}, u_{k2}, \dots, u_{kn})^{\mathrm{T}}$. From (3.45), the condition (1.4) is equivalent to

$$\sigma^h_{\min} = \min_k \sigma_k \geq c_0 > 0, \tag{3.46}$$

which is also equivalent to $\hat{\lambda}^h_{\min} \neq 0$ based on Lemma 3.3. From the above arguments we can conclude that (1.4) is equivalent to the uniqueness solution. Since (1.4) is complicated and difficult for application of 2D problems, we may use (3.46) instead. Note that both (1.4) and (3.46) are independent of f. However, (3.46) or $|\bar{\lambda}^h_{\min}| \geq c_0 > 0$ is, indeed, relevant to the numerical eigenvalues.

4. Numerical results

We consider the simple equation,

$$-\Delta u - z_p^2 u = f \quad \text{in } S, \quad u = 0 \text{ in } \Gamma, \tag{4.1}$$

where $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, $S = \{(x, y) | -1 < x < 1,\ \ 0 < y < 1\}$, and $z_p \geq 0$ is a parameter. The corresponding eigenvalue problem is given by

$$-\Delta w - z_p^2 w = \lambda w \quad \text{in } S, \quad w = 0 \text{ in } \Gamma. \tag{4.2}$$

Since the eigenfunctions of (4.2) are known as

$$w_{k,l} = \frac{1}{\sqrt{2}} \sin \frac{1+x}{2} k\pi \ \sin l\pi y, \tag{4.3}$$

we have the eigenvalues,

$$\lambda_{k,l} = \left(\frac{k}{2}\pi\right)^2 + (l\pi)^2 - z_p^2. \tag{4.4}$$

In the computation, choose $z_p^2 \approx \lambda_{1,1}$. Then the minimal eigenvalue is obtained by

$$\lambda_{\min} = \lambda_{1,1} - z_p^2 = \frac{5}{4}\pi^2 - z_p^2, \tag{4.5}$$

and the corresponding eigenfunction by

$$w_{\min} = w_{1,1} = \frac{1}{\sqrt{2}} \sin\left\{\frac{(1+x)\pi}{2}\right\} \sin \pi y. \tag{4.6}$$

The uniform difference grids are used for the FDM with the mesh size $h = \frac{1}{N}$. The standard five nodes schemes are used for (4.1),

$$(4 - z_p^2 h^2) u_{i,j} - \{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\} = h^2 f_{i,j}, \quad (i,j) \in S, \tag{4.7}$$
$$u_{i,j} = 0 \quad \text{in } \Gamma,$$

where $u_{i,j}$ denote the difference approximation of the true solution $u(x_i, y_j)$. Choose $f = 1$ in (4.1). The division numbers of S along x and y are $2N$ and N, respectively. We choose $N = 2, 4, 8, 16, 32$. In computation, let $z_p = \pi$, $\frac{\sqrt{5}}{2}\pi$ and $\frac{\sqrt{5}}{2}\pi \times 0.99$, which give from (4.5) the minimal eigenvalues as

$$\lambda_{\min} = \frac{1}{4}\pi^2, 0, 0.025\pi^2,$$

respectively. The computed results are listed in Tables 4.1–4.3, where the condition number is defined by

$$\text{Cond} = \frac{\lambda^h_{\max}}{\lambda^h_{\min}}, \tag{4.8}$$

Table 4.2
The numerical results by the FDM with $z_p = \frac{\sqrt{5}}{2}\pi$ and $f = 1$.

N	Av	$Av2$	$u_h(0, 0.5)$	$u_h(0.5, 0.5)$	$\lambda^h_{\min}$	$\lambda^h_{\max}$	Cond
2	3.58	3.17	−4.65	−3.05	−0.498	2.87	−5.75
4	12.2	12.3	−14.9	−21.5	−0.330(−1)	6.54	−198
8	42.8	47.1	−88.3	−62.2	−0.210(−2)	7.62	−0.364(4)
16	158	183	−355	−251	0.131(−3)	7.89	0.600(5)
32	604	723	−1443	−1006	0.822(−5)	7.94	0.965(6)

Table 4.3
The numerical results by the FDM with $z_p = \frac{\sqrt{5}}{2}\pi \times 0.99$ and $f = 1$.

N	Av	$Av2$	$u_h(0, 0.5)$	$u_h(0.5, 0.5)$	$\lambda^h_{\min}$	$\lambda^h_{\max}$	Cond
2	4.23	3.74	−5.45	−3.62	−0.437	2.91	−6.67
4	23.6	23.7	−41.2	−28.9	−0.177(−1)	6.55	−371
8	53.3	58.5	109	77.5	0.174(−2)	7.63	0.438(4)
16	26.0	30.1	58.1	41.3	0.828(−3)	7.89	0.954(4)
32	22.2	26.5	52.0	37.0	0.232(−3)	7.94	0.343(5)

Table 4.4
The numerical results by the FEM with $z_p = \pi$ and $f = 1$.

N	Av	$Av2$	$u_h(0, 0.5)$	$u_h(0.5, 0.5)$	$\lambda^h_{\min}$	$\lambda^h_{\max}$	Cond
2	3.40	2.97	3.98	3.11	0.711	2.60	3.66
4	3.31	3.26	5.39	4.08	0.166	3.50	21.1
8	3.09	3.34	6.00	4.50	0.394(−1)	3.86	98.1
16	2.91	3.32	6.18	4.62	0.969(−2)	3.96	408
32	2.80	3.29	6.22	4.66	0.241(−2)	3.98	0.165(4)

and $\lambda^h_{\max}$ and $\lambda^h_{\min}$ are the maximal and the minimal eigenvalues of the discrete matrix from (4.7), respectively. In the tables, $u_h(0, 0.5)$ and $u_h(0.5, 0.5)$ are the approximate solutions. From Table 4.1, we can see

$$\begin{aligned} &Av \le C, \qquad Av2 \le C, \\ &|u_h(0, 0.5)| \le C, \qquad |u_h(0.5, 0.5)| \le C, \\ &\lambda^h_{\min} = O(h^2), \qquad \lambda^h_{\max} = O(1), \quad \text{Cond} = O\left(\frac{1}{h^2}\right), \end{aligned} \tag{4.9}$$

and from Table 4.2,

$$\begin{aligned} &Av = O\left(\frac{1}{h^2}\right), \qquad Av2 = O\left(\frac{1}{h^2}\right), \\ &|u_h(0, 0.5)| = O\left(\frac{1}{h^2}\right), \qquad |u_h(0.5, 0.5)| = O\left(\frac{1}{h^2}\right), \\ &\lambda^h_{\min} = O(h^4), \qquad \lambda^h_{\max} = O(1), \qquad \text{Cond} = O\left(\frac{1}{h^4}\right). \end{aligned} \tag{4.10}$$

For $\lambda_{\min} \neq 0$, the boundedness of Av and $Av2$ is proved in Section 2, which is confirmed by (4.9). On the other hand, for $\lambda_{\min} = 0$ but $\lambda^h_{\min} \neq 0$, the Av and the $Av2$ are unbounded as (2.38). This conclusion is supported by (4.10) numerically. In this case, the solution obtained from (4.7) is given by $\boldsymbol{x_h} = \frac{\alpha^h_1}{\lambda^h_{\min}} \boldsymbol{y^h_{min}} (\asymp \frac{\alpha^h_1}{h^2} \boldsymbol{y^h_{min}})$, which is called the ghost solution in [1].

When $z_p^2 \neq \lambda_{\min}$ but $z_p^2 \approx \lambda_{\min}$, the solutions of (4.1) exist uniquely. The boundedness of Av and $Av2$ can be observed only if h is small (i.e., N is large). From Table 4.3 with $z_p = \frac{\sqrt{5}}{2}\pi \times 0.99$, we can see the bounded values Av and $Av2$ for $N \ge 16$. This numerical result can be explained by (2.22).

For the FEM, we choose the uniform bilinear FEM, and the computed results are listed in Tables 4.4–4.6. The boundedness and the unboundedness of Av, $Av2$, $u_h(0, 0.5)$ and $u_h(0.5, 0.5)$ can be seen from Tables 4.4 and 4.5, respectively. When $N \ge 16$ we can also see in Table 4.6 the boundedness of Av, $Av2$, $u_h(0, 0.5)$ and $u_h(0.5, 0.5)$. Hence, all the numerical conclusions by the FEM are exactly the same as those observed from Tables 4.1–4.3 by the FDM. All tables in this paper are obtained from Fortran programs in double precision.

Finally, choose the solution in (4.1),

$$u = u(x, y) = \sin \pi x \sin \pi y. \tag{4.11}$$

Table 4.5
The numerical results by the FEM with $z_p = \frac{\sqrt{5}}{2}\pi$ and $f = 1$.

N	Av	$Av2$	$u_h(0, 0.5)$	$u_h(0.5, 0.5)$	$\lambda^h_{\min}$	$\lambda^h_{\max}$	Cond
2	5.31	4.65	6.40	4.77	0.340	2.48	7.30
4	13.5	13.4	23.1	16.5	0.302(−1)	3.45	114
8	43.8	48.1	89.8	63.7	0.205(−2)	3.85	0.188(4)
16	159	184	357	253	0.131(−3)	3.96	0.303(5)
32	605	724	1426	1008	0.821(−5)	3.98	0.485(6)

Then $f(x, y) = (2\pi^2 - z_p^2)u$. We set $z_p = \frac{\sqrt{5}}{2}\pi$ to get $\lambda_{\min} = 0$. The computed results by the FDM and the FEM are listed in Tables 4.7 and 4.8, respectively. From Table 4.7, we can see

$$Av \le C, \qquad Av2 \le C, \qquad |u_h(0.5, 0.5)| \le C, \tag{4.12}$$

$$\max |\epsilon_{i,j}| = O(h^2), \qquad \max |D\epsilon_{i,j}| = O(h^2), \tag{4.13}$$

$$\overline{\|\epsilon\|}_0 = O(h^2), \qquad \overline{\|\epsilon\|}_1 = O(h^2), \tag{4.14}$$

where $\epsilon = u - u_h$, $\max |D\epsilon_{ij}| = \max_{i,j}\{|\frac{\partial \epsilon_{i,j}}{\partial x}|, \ |\frac{\partial \epsilon_{i,j}}{\partial y}|\}$, and $\overline{\|\epsilon\|}_k$ are the discrete H^k- norms defined in Li et al. [9]. From Table 4.8 by the bilinear FEM, the same bounds and asymptotics as (4.12)–(4.14) are found, except $\overline{\|\epsilon\|}_k$ are replaced by the H^k norms $\|\epsilon\|_{k,S}$. From (4.12)–(4.14), there arise questions: Why have the unboundedness of Av, $Av2$ and $u_h(0.5, 0.5)$ not been observed for $N \le 32$? How can we explain the optimal convergence rates in (4.13) and (4.14) at $\lambda_{\min} = 0$? The reason is that the orthogonality (2.40) happens:

$$\begin{aligned}\alpha_1 = \iint_S f w_{1,1} &= \frac{3\pi^2}{4\sqrt{2}} \int_0^1 (\sin \pi y)^2 \mathrm{d}y \times \int_{-1}^1 \sin \pi x \sin \frac{1}{2}\pi(1 + x)\mathrm{d}x \\ &= -\frac{3\pi^2}{8\sqrt{2}} \int_{-1}^1 \sin \pi x \cos \frac{\pi}{2} x \mathrm{d}x = 0,\end{aligned} \tag{4.15}$$

where we have used the equality,

$$\int_{-1}^0 \sin \pi x \cos \frac{\pi}{2} x \mathrm{d}x = -\int_0^1 \sin \pi x \cos \frac{\pi}{2} x \mathrm{d}x. \tag{4.16}$$

For $\lambda_{\min} = 0$, we have $\lambda^h_{\min} \asymp O(h^2)$ as in (2.32) due to discrete and rounding errors. The coefficient is given by

$$\alpha_1^h = \langle \boldsymbol{B_h F_h}, \boldsymbol{y^h_{min}} \rangle = 0, \tag{4.17}$$

since the discrete form of (4.16) also retains. However, the real $\alpha_1^h \asymp O(\tau)$, where τ is the rounding error of computer, and $\tau = O(10^{-17})$ for double precision. Hence we have the solution from (2.20)

$$\boldsymbol{x_h} = \frac{\alpha_1^h}{\lambda^h_{\min}} \boldsymbol{y^h_{min}} + \frac{\alpha_2^h}{\lambda^h_{next-min}} \boldsymbol{y^h_{next-min}} + \cdots, \tag{4.18}$$

where $\lambda_{next-min}$ is the next minimal eigenvalue in magnitude. The expansion coefficient α_1^h from rounding errors is so small that the condition (2.24) is violated, and it gives

$$\left| \frac{\alpha_1^h}{\lambda^h_{\min}} \right| < \left| \frac{\alpha_2^h}{\lambda^h_{next-min}} \right|.$$

Then solution (4.18) leads to

$$\boldsymbol{x_h} \approx \frac{\alpha_2^h}{\lambda^h_{next-min}} \boldsymbol{y^h_{next-min}}. \tag{4.19}$$

Hence, the unboundedness does not show up. A similar example is also found in Hu and Li [10]. This numerical sample displays the pitfall to determine the uniqueness of solutions, purely based on observation of the numerical solutions, since the infinitesimal h can not be chosen in real computation. The strict judgment of the uniqueness solution is $\lambda_{\min} \ne 0$, which should be solicited by the eigenvalue problem (1.5). For $\lambda_{\min} = 0$ or $\lambda_{\min} \approx 0$, since the accuracy of numerical eigenvalues is crucial, the spectral and Trefftz methods are suggested due to exponential convergence rates, see [11,4,12].

To close this paper, let us make a few final remarks.

Table 4.6
The numerical results by the FEM with $z_p = \frac{\sqrt{5}}{2}\pi \times 1.01$ and $f = 1$.

N	Av	$Av2$	$u_h(0, 0.5)$	$u_h(0.5, 0.5)$	$\lambda^h_{\min}$	$\lambda^h_{\max}$	Cond
2	5.76	5.05	6.97	5.16	0.303	2.47	8.16
4	23.7	23.7	40.8	29.1	0.165(−1)	3.44	208
8	50.8	56.0	−105	−74.0	−0.170(−2)	3.85	−0.226(4)
16	24.1	28.0	−54.6	−38.4	−0.830(−3)	3.96	−0.476(4)
32	20.5	24.6	−48.7	−34.2	−0.233(−3)	3.98	−0.171(5)

Table 4.7
The numerical results by the FDM with $z_p = \frac{\sqrt{5}}{2}\pi$ and $f = \sin \pi x \sin \pi y$, where $\lambda_{\max}$, $\lambda_{\min}$ and Cond are the same as those in Table 4.2.

N	Av	$Av2$	$u_h(0.5, 0.5)$	$\max\lvert\epsilon_{ij}\rvert$	$\max\lvert D\epsilon_{ij}\rvert$	$\overline{\lVert\epsilon\rVert}_0$	$\overline{\lVert\epsilon\rVert}_1$
2	1.35	1.43	0.000	1.02	1.82	0.722	2.67
4	0.641	0.667	0.817	1.55	0.364	0.110	0.410
8	0.498	0.553	0.956	0.353(−1)	0.883(−1)	0.250(−1)	0.934(−1)
16	0.447	0.521	0.989	0.863(−2)	0.219(−1)	0.610(−2)	0.229(−1)
32	0.425	0.509	0.997	0.215(−2)	0.547(−2)	0.152(−2)	0.568(−2)

Table 4.8
The numerical results by the FEM with $z_p = \frac{\sqrt{5}}{2}\pi$ and $f = \sin \pi x \sin \pi y$, where $\lambda_{\max}$, $\lambda_{\min}$ and Cond are the same as those in Table 4.5.

N	Av	$Av2$	$u_h(0.5, 0.5)$	$\max\lvert\epsilon_{ij}\rvert$	$\max\lvert D\epsilon_{ij}\rvert$	$\lVert\epsilon\rVert_{0,S}$	$\lVert\epsilon\rVert_{1,S}$
2	4.23	4.49	0.635	0.365	0.952	0.258	1.37
4	0.487	0.506	0.877	0.123	0.421	0.887(−1)	0.464
8	0.465	0.517	0.967	0.333(−1)	0.122	0.235(−1)	0.126
16	0.440	0.512	0.991	0.851(−2)	0.316(−1)	0.601(−2)	0.323(−1)
32	0.423	0.507	0.998	0.213(−2)	0.797(−2)	0.151(−2)	0.812(−2)

1. In Section 2, we consider the second order self-adjoint boundary value problems in 2D and develop the study of the uniqueness of solutions of two-point boundary value problems in [2]. We obtain the better uniqueness conditions (1.15), (1.18) and (1.19), which are simple and easy in real computations. In Section 3, the uniqueness conditions (1.15), (1.18) and (1.19) are discussed for non-self-adjoint elliptic problems. The results in this paper can also be extended to higher order elliptic boundary value problems of 2D and 3D. An application of these conditions is that we may discover the uniqueness simultaneously, while seeking the approximate solutions of elliptic boundary problems.
2. For simplicity, we only consider the linear and bilinear elements in this paper. The results of this paper can be extended to the higher order FEM, including the nonconforming elements. For the nonconforming lower elements, since the similar errors bounds as Lemma 2.1 can be found in [13], the results in this paper also hold.
3. Care must be taken for the case when the operator $\mathcal{L}$ in (1.1) is nearly singular (i.e., $\lambda_{\min} \approx 0$). In this case, the h should be smaller, see Tables 4.3 and 4.6. In particular, if the given $f(x, y)$ is just or nearly orthogonal to the eigenfunction $w_{\min}$ of the zero eigenvalue $\lambda_{\min} = 0$, the misleading conclusions may be drawn, purely from numerical observation, because the behavior of solutions and errors is just like that of $\lambda_{\min} \neq 0$, see Tables 4.7 and 4.8. To avoid such a pitfall, we should choose different functions $f(x, y)$ for the same FEM (or FDM), use the h as small as possible, and scrutinize the behavior of solutions and errors.
4. When $\lambda_{\min} \approx 0$ the degeneracy of the Helmholtz equation is studied, and the error bounds with $O(\frac{1}{|\lambda_{\min}|})$ are derived for the Trefftz method in Li [11,4,12]. When $\lambda_{\min} \approx 0$, the solution errors from the FDM and the FEM may have the similar bounds with $O(\frac{1}{|\lambda_{\min}|})$. Hence the constant C depends on $\frac{1}{|\lambda_{\min}|}$.
5. The uniqueness of solutions of (1.1) is equivalent to $\lambda_{\min} \neq 0$ of the corresponding eigenvalue problem (1.5). When $\lambda_{\min} = 0$ or $\lambda_{\min} \approx 0$, highly accurate algorithms, such as the spectral and Trefftz methods in [11,4,12], are recommended to study the problem. Such a suggestion is also valid for the non-self-adjoint problem of (3.1).
6. In matrix analysis, a number of equivalent conditions of the nonsingularity matrix are discussed in Horn and Johnson [14]. In this paper, we explore the simple equivalent conditions for the uniqueness solutions of numerical partial differential equations, which may be easily carried out during the approximate procedure. When we solve the linear algebraic equations (1.11), where the nonsingularity of matrix $\boldsymbol{A}$ is unknown, the conditions (1.15), (1.18) and (1.19) can also be regarded as the equivalent conditions of nonsingularity. Hence this paper may enrich the matrix analysis.

Acknowledgments

The authors are very grateful to Prof. Tetsuro Yamamoto for his valuable comments and suggestions through e-mail communication. The authors also thank the referee for his careful reading and helpful suggestions. The second author is supported by Scientific Research Grant-in-Aid from JSPS under grant 21540106. The fourth author is supported by the National Natural Science Foundation of China under grant 10871051, Shanghai Municipal Science and Technology Committee under grant 08511501703, Shanghai Municipal Education Committee (Dawn Project) and 973 Program Project (No. 2010CB327900).

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